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Smooth projective surfaces with pseudo-effective tangent bundles

joint work with Yongnam Lee and Guolei Zhong

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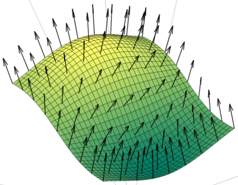

Positivity of Vector Bundles

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Let X be a smooth projective variety.

Given a vector bundle $\mathcal{E}$ on X,
denote by $\mathbb{P}(\mathcal{E}) := \mathbb{P}(\mathrm{Sym}^\bullet \mathcal{E})$ the Grothendieck projectivization of $\mathcal{E}$
with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the relative hyperplane section bundle.

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Definition

The vector bundle $\mathcal{E}$ on X is ample (resp. nef, big, pseudo-effective)
if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample (resp. nef, big, pseudo-effective) on $\mathbb{P}(\mathcal{E})$.

Remark

Let $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ be the projection. Then $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m) = \operatorname{Sym}^m \mathcal{E}$ for all $m \geq 0$.

Theorem (Hartshorne's Conjecture, Mori, 1979)

Let X be a smooth projective variety and let $\mathcal{T}_X$ be its tangent bundle. Then $\mathcal{T}_X$ is ample if and only if $X \simeq \mathbb{P}^n$.

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Theorem (Mallory, 2021 and Miyaoka, 1987)

Let X be a smooth projective variety. If $\mathcal{T}_X$ is big then X is uniruled.

Conjecture (Campana-Peternell, 1991)

Any Fano manifold whose tangent bundle is nef is rational homogeneous.

Theorem (Demailly-Peternell-Schneider, 1994)

Any compact Kähler manifold with nef tangent bundle admits a finite étale cover with smooth Albanese map whose fibres are Fano manifolds with nef tangent bundle.

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- $\mathcal{T}_X$ *is pseudo-effective if and only if* $d \geq 4$.
- $\mathcal{T}_X$ *is big if and only if* $d \geq 5$.

Moreover $\mathcal{T}_X$ *is pseudo-effective if and only if* $H^0(X, \mathrm{Sym}^m \mathcal{T}_X) \neq 0$ *for some* $m \in \mathbb{N}$.

Let X be a smooth projective variety.
Recall the augmented irregularity $q^\circ(X)$ of X is defined to be the supremum of

$$q(X') := h^1(X', \mathcal{O}_{X'})$$

where $X' \to X$ runs over all the finite étale covers of X (Nakayama-Zhang, 2009).

Question

Let X be a non-uniruled smooth projective variety of dimension n.
Are the following assertions equivalent?

1. The tangent bundle $\mathcal{T}_X$ is pseudo-effective;
2. The top Chern class $c_n(X)$ vanishes, and the augmented irregularity $q^\circ(X)$ does not vanish.

Lemma (Höring-Liu-Shao, 2022 and Druel, 2018)

Let X be a projective variety, $\mathcal{E}$ a vector bundle on X, and H a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then $\mathcal{E}$ is pseudo-effective if and only if for all $c > 0$ there exist sufficiently divisible integers $i, j \in \mathbb{N}$ such that $i > cj$ and

$$H^0(X, \operatorname{Sym}^i \mathcal{E} \otimes \mathcal{O}_X(jH)) \neq 0.$$

Note that $H^0(\mathbb{P}_X(\mathcal{E}), \mathcal{O}(i\xi + j\pi^* H)) = H^0(X, \operatorname{Sym}^i \mathcal{E} \otimes \mathcal{O}_X(jH))$.

Corollary

Let $\mathcal{E} \subseteq \mathcal{F}$ be an injection between two vector bundles over a projective variety X. If $\mathcal{E}$ is pseudo-effective, then so is $\mathcal{F}$.

Remark

The quotient bundle of a pseudo-effective vector bundle may not be pseudo-effective.

Descending of pseudo-effectivity

Corollary (Höring-Liu-Shao, 2022)

Let $\pi \colon X' \to X$ be a birational morphism between smooth projective varieties. If the tangent bundle $\mathcal{T}_{X'}$ is pseudo-effective, then so is $\mathcal{T}_X$.

Sketch of Proof.

Let $Z \subset X$ be the image of the exceptional locus.
For every $i \in \mathbb{N}$, $\pi_*(\operatorname{Sym}^i \mathcal{T}_{X'})$ is torsion-free and $\operatorname{Sym}^i \mathcal{T}_X$ is reflexive.
Since $\pi_*(\operatorname{Sym}^i \mathcal{T}_{X'}) = \operatorname{Sym}^i \mathcal{T}_X$ on $X \setminus Z$ and $\operatorname{codim}_X Z \geq 2$, there is an injection

$$\pi_*(\operatorname{Sym}^i \mathcal{T}_{X'}) \hookrightarrow \operatorname{Sym}^i \mathcal{T}_X. \qquad \square$$

Theorem (Höring-Peternell, 2019)

Let X be a normal projective variety with at most klt singularities such that $K_X \equiv 0$. Suppose that X is smooth in codimension two. If the reflexive cotangent sheaf $\Omega_X^{[1]}$ or the tangent sheaf $\mathcal{T}_X$ is pseudo-effective, then $q^\circ(X) \neq 0$.

Corollary (Höring-Peternell, 2019 and Nakayama, 2004 and ...)

If X is a (singular) Calabi–Yau or irreducible symplectic variety that is smooth in codimension two, then $\Omega_X^{[1]}$ and $\mathcal{T}_X$ are not pseudo-effective.

Proposition (Höring-Peternell, 2021)

Let X be a projective manifold such that $\mathcal{T}_X$ is pseudo-effective. If X is not uniruled, there exists exists a decomposition

$$\mathcal{T}_X \simeq \mathcal{F} \oplus \mathcal{G},$$

where $\mathcal{F} \neq 0$ and $\mathcal{G}$ are integrable subbundles such that $c_1(\mathcal{F}) = 0$.

Corollary

The X above is NOT of general type.

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Moreover, if one of the above equivalent conditions holds, then

- the Kodaira dimension $\kappa(\mathbb{P}(\mathcal{T}_S), \mathcal{O}(1)) = 1 - \kappa(S) \in \{0, 1\}$, and

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Moreover, if one of the above equivalent conditions holds, then

- the Kodaira dimension $\kappa(\mathbb{P}(\mathcal{T}_S), \mathcal{O}(1)) = 1 - \kappa(S) \in \{0, 1\}$, and
- there is a finite étale cover $S' \to S$ such that S' is either an abelian surface or a product $E \times F$ where E is an elliptic curve and F is smooth of genus ≥ 2.

Counter-example: Higher Dimension

Example

Let S be a non-minimal smooth projective surface S which contains some (-1)-curve. Let $X := E \times S$ where E is an elliptic curve and $p\colon X \to E$ the projection.

Considering the natural injection

$$0 \longrightarrow p^*\mathcal{O}_E \longrightarrow \mathcal{T}_X,$$

then $\mathcal{T}_X$ is pseudo-effective. However, K_X is not nef.

Uniruled Ones

Let X be a projective variety of dimension n, H a nef and big divisor on X, and $\mathcal{F}$ a torsion free coherent sheaf on X.

Definition

The slope of $\mathcal{F}$ with respect to H is defined to be the rational number

$$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\operatorname{rank}(\mathcal{F})}$$

where c_1 is the first Chern class.

Definition

A *torsion-free* coherent sheaf $\mathcal{E}$ is said to be μ-semi-stable if for any non-zero subsheaf $\mathcal{F} \subseteq \mathcal{E}$, the slopes satisfy the inequality $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$.

Theorem (Kim, 2022)

Let C be a smooth projective curve and $\mathcal{E}$ a vector bundle on C. Then the projective bundle $X = \mathbb{P}_C(\mathcal{E})$ has big tangent bundle $\mathcal{T}_X$ if and only if $\mathcal{E}$ is unstable or $C = \mathbb{P}^1$.

Proposition

The tangent bundle $\mathcal{T}_X$ of any projective bundle $f\colon X = \mathbb{P}_C(\mathcal{E}) \to C$ over a smooth curve C is pseudo-effective.
In particular, $\mathcal{T}_X$ is pseudo-effective but non-big iff $\mathcal{E}$ is semi-stable and $C \not\simeq \mathbb{P}^1$.

Proof.
NTS: $\mathcal{E}$ being semi-stable implies $\mathcal{T}_X$ is pseudo-effective.
Since the determinant $\det(\mathcal{E}^\vee \otimes \mathcal{E}) \simeq \mathcal{O}_C$, the semi-stable vector bundle $\mathcal{E}^\vee \otimes \mathcal{E}$ is nef.
Consider the following relative Euler sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f^*\mathcal{E}^\vee \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{T}_{X/C} \longrightarrow 0,$$

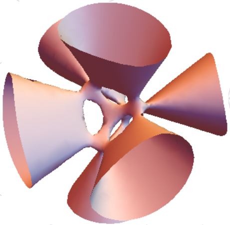
where the relative tangent bundle $\mathcal{T}_{X/C} := \Omega^\vee_{X/C}$. Then the following composite map

$$f^*(\mathcal{E}^\vee \otimes \mathcal{E}) \longrightarrow f^*\mathcal{E}^\vee \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{T}_{X/C}$$

is a surjection, which implies that $\mathcal{T}_{X/C}$ is nef and hence pseudo-effective.

Proposition

Let $f : S = \mathbb{P}_C(\mathcal{E}) \to C$ be a $\mathbb{P}^1$-bundle over a smooth non-rational curve C. Suppose the tangent bundle $\mathcal{T}_S$ is pseudo-effective but not big. Then the blow-up of S along a point p has pseudo-effective tangent bundle if and only if there exist some positive integer m and some line bundle $\mathcal{L} \equiv \mathcal{T}_{S/C}$ such that $H^0(S, \mathfrak{m}_p^m \otimes \mathcal{L}^{\otimes m}) \neq 0$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{S,p}$.



Non-uniruled Surfaces

Proposition

Let S be a smooth projective surface of $\kappa(S) = 0$.
If $\mathcal{T}_S$ is pseudo-effective, then K_S is nef, i.e., S is minimal.

Remark ($\kappa = 0$ & minimal)

Enriques surface, K3 surface, bi-elliptic surface or abelian surface

Lemma

Let S be a smooth minimal projective surface with $\kappa(S) = 0$.
The tangent bundle $\mathcal{T}_S$ is pseudo-effective if and only if S is a Q-abelian surface

Proposition

Let S be a smooth projective surface of general type.
Then $\mathcal{T}_S$ is not pseudo-effective.

By the semi-stability of the tangent bundle $\mathcal{T}_S$ with respect to the nef and big K_S, one has $H^0(S, \mathrm{Sym}^i \mathcal{T}_S \otimes \mathcal{O}_S(jK_S)) = 0$.

Remark

Let X be a normal ($\mathbb{Q}$-factorial) projective variety
which is of general type and has at worst klt singularities.
The reflexive tangent sheaf $\mathcal{T}_X := \Omega_X^\vee$ is not pseudo-effective [Höring-Peternell, 2020].

Example: blow-up of a non-reduced centre

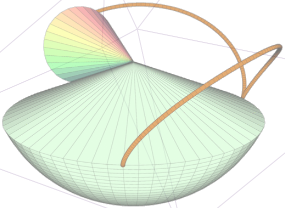
Let $\mathcal{I} = (x^2, y^2) \subseteq \mathbb{C}[x,y]$ be an ideal and let $\pi\colon X = \mathrm{Bl}_{\mathcal{I}}(\mathbb{A}^2) \to \mathbb{A}^2$ be the blow-up. Denote by L the line $y = x$ on $\mathbb{A}^2$, $\widetilde{L}$ its proper transform and E the exceptional divisor.

Claim: $\pi^* L = \widetilde{L} + 1/2E$.

Let $[a:b]$ be the homogeneous coordinates of $\mathbb{P}^1$.
Then X is defined by $y^2 a - x^2 b = 0$ in $\mathbb{A}^2 \times \mathbb{P}^1$.

- $\widetilde{L}.E = 2$;
- $E^2 = -4$, which is the (Samuel) multiplicity of the blown-up point;
- E is defined by

$$\begin{cases} y^2 a - x^2 b = 0, \\ x^2 = 0, \\ y^2 = 0, \end{cases} \quad \text{or just} \quad \begin{cases} y^2 a - x^2 b = 0, \\ x^2 = 0, \end{cases} \quad \text{on the affine chart } a \neq 0.$$



Thanks for your attention!