Jia Jia

Tsinghua University, China

August 17, 2023

Wild Automorphisms of Compact Complex Spaces

of lower dimensions

joint work with Long Wang

Wild Automorphism

Definition

Let X be a compact complex space.

We will use the analytic Zariski topology on X whose closed sets are all analytic sets.

Let X be a compact complex space.
We will use the analytic Zariski topology on X whose closed sets are all analytic sets.

Definition

An automorphism $\sigma \in \mathrm{Aut}(X)$ is called wild in the sense of Reichstein-Rogalski-Zhang if for any non-empty analytic subset Z of X satisfying $\sigma(Z) = Z$, we have $Z = X$; or equivalently, for every point $x \in X$, its orbit $\{\sigma^n(x) \mid n \geq 0\}$ is Zariski dense in X.

Lemma 1

Let X be a compact complex space and let σ be an automorphism on X.

Lemma 1

Let X be a compact complex space and let σ be an automorphism on X.

1. If σ is wild then X is smooth.

Lemma 1

Let X be a compact complex space and let σ be an automorphism on X.

1. If σ is wild then X is smooth.
2. σ is wild if and only if σ^m is wild for some $m \geq 1$ (and hence for all $m \geq 1$). In particular, a wild automorphism has infinite order.

Lemma 1

Let X be a compact complex space and let σ be an automorphism on X.

1. If σ is wild then X is smooth.
2. σ is wild if and only if σ^m is wild for some $m \geq 1$ (and hence for all $m \geq 1$). In particular, a wild automorphism has infinite order.

Proof.

□

Lemma 1

Let X be a compact complex space and let σ be an automorphism on X.

1. If σ is wild then X is smooth.
2. σ is wild if and only if σ^m is wild for some $m \geq 1$ (and hence for all $m \geq 1$). In particular, a wild automorphism has infinite order.

Proof.

1. The singular locus $\operatorname{Sing} X$ is an analytic subset of X and stabilised by every automorphism.

$\square$

Lemma 1

Let X be a compact complex space and let σ be an automorphism on X.

1. If σ is wild then X is smooth.
2. σ is wild if and only if σ^m is wild for some $m \geq 1$ (and hence for all $m \geq 1$). In particular, a wild automorphism has infinite order.

Proof.

1. The singular locus $\mathrm{Sing}\, X$ is an analytic subset of X and stabilised by every automorphism.
2. If σ^m stabilised an analytic subset Z of X, then σ stabilises the analytic subset $\bigcup_{i=0}^{m-1} \sigma^i(Z)$ of X.

□

Proposition 2

Let X be a compact complex manifold with a wild automorphism σ.

Proposition 2

Let X be a compact complex manifold with a wild automorphism σ.

1. The Euler-Poincaré characteristic $\chi(\mathcal{O}_X) = 0$, and the topological Euler number $e(X) = 0$.
   In particular, X is not rationally connected.

Proposition 2

Let X be a compact complex manifold with a wild automorphism σ.

1. The Euler-Poincaré characteristic $\chi(\mathcal{O}_X) = 0$, and the topological Euler number $e(X) = 0$.
   In particular, X is not rationally connected.
2. Let D be a Cartier divisor on X such that $\sigma^* D \sim D$. Then either $|D| = \emptyset$ or $D \sim 0$.
   In particular, the Kodaira dimension $\kappa(X) \leq 0$; if $\kappa(X) = 0$, then $K_X \sim_{\mathbb{Q}} 0$.

Proposition 2

Let X be a compact complex manifold with a wild automorphism σ.

1. The Euler-Poincaré characteristic $\chi(\mathcal{O}_X) = 0$, and the topological Euler number $e(X) = 0$.
   In particular, X is not rationally connected.
2. Let D be a Cartier divisor on X such that $\sigma^* D \sim D$. Then either $|D| = \emptyset$ or $D \sim 0$.
   In particular, the Kodaira dimension $\kappa(X) \leq 0$; if $\kappa(X) = 0$, then $K_X \sim_{\mathbb{Q}} 0$.
3. Suppose that X is Kähler and $\kappa(X) = 0$. Then the Beauville-Bogomolov (minimal split) finite étale cover $\widetilde{X}$ of X is a product of a complex torus T and some copies of Calabi-Yau manifolds C_i in the strict sense; a positive power of σ lifts to a diagonal action on $\widetilde{X} = T \times \prod_i C_i$ whose action on each factor is wild.

Let X be a compact Kähler manifold of dimension $n \geq 1$, and $f \in \mathrm{Aut}(X)$.
Denote by $d_i(f)$ the i-th dynamical degree of f, that is, the spectral radius of $f^*|_{H^{i,i}(X)}$.

Entropy and Dynamical Degrees

Let X be a compact Kähler manifold of dimension $n \geq 1$, and $f \in \mathrm{Aut}(X)$.
Denote by $d_i(f)$ the i-th dynamical degree of f, that is, the spectral radius of $f^*|_{H^{i,i}(X)}$.

The dynamical degrees are log concave, i.e., $i \mapsto \log d_i(f)$ is concave for $1 \leq i \leq n-1$.
That is $d_{i-1}(f) d_{i+1}(f) \leq d_i(f)^2$ for all $1 \leq i \leq n-1$.
Hence $d_i(f) = 1$ for one i with $1 \leq i \leq n-1$ implies that it holds for all such i.

Entropy and Dynamical Degrees

Let X be a compact Kähler manifold of dimension $n \geq 1$, and $f \in \mathrm{Aut}(X)$.
Denote by $d_i(f)$ the i-th dynamical degree of f, that is, the spectral radius of $f^*|_{H^{i,i}(X)}$.

The dynamical degrees are log concave, i.e., $i \mapsto \log d_i(f)$ is concave for $1 \leq i \leq n-1$.
That is $d_{i-1}(f)d_{i+1}(f) \leq d_i(f)^2$ for all $1 \leq i \leq n-1$.
Hence $d_i(f) = 1$ for one i with $1 \leq i \leq n-1$ implies that it holds for all such i.

The topological entropy $h(f)$ of a map f is a dynamical invariant.
The classical results of Gromov-Yomdin imply that

$$h(f) = \log \max_{1 \leq i \leq n} \{d_i(f)\}.$$

Hence f has zero entropy if and only $d_1(f) = 1$.

Conjecture Reichstein-Rogalski-Zhang 2006

Assume that a compact Kähler space X admits a wild automorphism.
Then X is isomorphic to a complex torus.

Conjecture Reichstein-Rogalski-Zhang 2006

Assume that a compact Kähler space X admits a wild automorphism.
Then X is isomorphic to a complex torus.

Remark

This is not true if we remove the Kähler condition.

Conjectures

Conjecture Reichstein-Rogalski-Zhang 2006

Assume that a compact Kähler space X admits a wild automorphism.
Then X is isomorphic to a complex torus.

Remark

This is not true if we remove the Kähler condition.

The following conjecture is a little bit weaker.

Conjecture Oguiso-Zhang 2022

Every wild automorphism σ of a compact Kähler space X has zero entropy.

Known Results

Theorem (Oguiso-Zhang 2022)

Let X be a projective variety over $\mathbb{C}$ of dimension ≤ 3. Assume that X admits a wild automorphism σ. Then either X is an abelian variety, or X is a Calabi Yau manifold of dimension three and σ has zero entropy.

Proposition 3

Let X be a complex torus and let σ be a wild automorphism on X.
Then σ has zero entropy.

Proposition 3

Let X be a complex torus and let σ be a wild automorphism on X.
Then σ has zero entropy.

Proof.

Write $\sigma = T_b \circ \alpha$ for some translation T_b and $\alpha \in \mathrm{End}(X)$.
Since σ is wild, it can be shown that α is unipotent.
Clearly T_b acts on $H^1(X, \mathbb{C})$ as an identity.
We claim that the action of the unipotent $\alpha \in \mathrm{End}(X)$ on $H^1(X, \mathbb{C})$ is also unipotent.
In fact, $\mathrm{End}(X)_{\mathbb{Q}} := \mathrm{End}(X) \otimes \mathbb{Q}$ is contained in $M_{2n}(\mathbb{Q})$,
and the homomorphism $\mathrm{End}(X)_{\mathbb{Q}} \to \mathrm{GL}(H^1(X, \mathbb{C}))$ preserves unipotency.
Therefore, $d_1(\sigma) = 1$ and σ has zero entropy. □

Q-torus

A compact complex space X is called a Q-torus if

- it has a complex torus T_1 as an étale finite cover; or equivalently,

Q-torus

A compact complex space X is called a Q-torus if

- it has a complex torus T_1 as an étale finite cover; or equivalently,
- it is the quotient of a complex torus T_2 by a finite group acting freely on T_2.

Q-torus

A compact complex space X is called a *Q-torus* if

- it has a complex torus T_1 as an étale finite cover; or equivalently,
- it is the quotient of a complex torus T_2 by a finite group acting freely on T_2.

Proposition 4

Let X be a Q-torus with a wild automorphism σ. Then X is a complex torus.

Q-torus

A compact complex space X is called a *Q*-torus if

- it has a complex torus T_1 as an étale finite cover; or equivalently,
- it is the quotient of a complex torus T_2 by a finite group acting freely on T_2.

Proposition 4

Let X be a *Q*-torus with a wild automorphism σ. Then X is a complex torus.

Proof.

Let $T \longrightarrow X$ be the minimal splitting cover of X.
Then σ lifts to an automorphism on T, also denoted as σ.
Note that the σ on T normalises $H := \mathrm{Gal}(T/X)$.
Hence $\sigma^{r!}$ centralises every element of H, where $r := |H|$.
Since $\sigma^{r!}$ is still wild, H consists of translations.
Hence $H = \{\mathrm{id}_T\}$ by the minimality of $T \longrightarrow X$.
Therefore, $X = T$ and X is a complex torus. □

Lemma 5

Let X be a compact complex space and let σ be an automorphism on X.

Lemma 5

Let X be a compact complex space and let σ be an automorphism on X.

1. Suppose that σ is wild and $f\colon X \longrightarrow Y$ (resp. $g\colon W \longrightarrow X$ with $g(\mathrm{Sing}(W)) \neq X$) is a σ-equivariant surjective morphism of compact complex spaces. Then f (resp. g) is a smooth morphism.

Lemma 5

Let X be a compact complex space and let σ be an automorphism on X.

1. Suppose that σ is wild and $f\colon X \longrightarrow Y$ (resp. $g\colon W \longrightarrow X$ with $g(\mathrm{Sing}(W)) \neq X$) is a σ-equivariant surjective morphism of compact complex spaces.
   Then f (resp. g) is a smooth morphism.
2. Suppose that $f\colon X \longrightarrow Y$ is a σ-equivariant surjective morphism to a compact complex space Y. If the action σ on X is wild then so is the action of σ on Y (and hence Y is smooth).

Lemma 5

Let X be a compact complex space and let σ be an automorphism on X.

1. Suppose that σ is wild and $f\colon X \longrightarrow Y$ (resp. $g\colon W \longrightarrow X$ with $g(\mathrm{Sing}(W)) \neq X$) is a σ-equivariant surjective morphism of compact complex spaces. Then f (resp. g) is a smooth morphism.
2. Suppose that $f\colon X \longrightarrow Y$ is a σ-equivariant surjective morphism to a compact complex space Y. If the action σ on X is wild then so is the action of σ on Y (and hence Y is smooth).
3. Suppose that $f\colon X \longrightarrow Y$ is a σ-equivariant generically finite surjective morphism of compact complex spaces. Then the action of σ on X is wild if and only if so is the action of σ on Y.
   Further, if this is the case, then $f\colon X \longrightarrow Y$ is a finite étale morphism, and in particular, it is an isomorphism when f is bimeromorphic.

A technical lemma

Lemma 6

Let X be a compact Kähler manifold with a wild automorphism σ, let A be a complex torus and let $f\colon X \longrightarrow A$ be a σ-equivariant surjective projective morphism with connected fibres of positive dimension. Assume general fibres of f are isomorphic to F. Suppose that a positive power σ_A^s of σ_A fixes some big $(1,1)$-class α on A in $H^{1,1}(A)$ (this holds if $\dim A = 1$ or a positive power of σ_A is a translation on A).
Then $-K_F$ is not a big divisor.

MRC

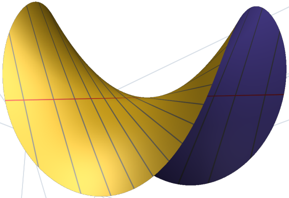
A maximal rational connected (MRC) fibration on a uniruled compact Kähler manifold has general fibres F rationally connected, and the base is not uniruled.

MRC

A maximal rational connected (MRC) fibration on a uniruled compact Kähler manifold has general fibres F rationally connected, and the base is not uniruled.

Lemma 7

Let X be a uniruled compact Kähler manifold of dimension ≥ 1, with a wild automorphism σ. Then we can choose the maximal rationally connected (MRC) fibration $X \longrightarrow Y$ to be a well-defined σ-equivariant surjective smooth morphism with $0 < \dim Y < \dim X$. Further, the action of σ on Y is also wild.



Curves and Surfaces

Theorem 8

Let X be a compact Kähler space of dimension ≤ 2. Assume that X admits a wild automorphism σ. Then X is a complex torus, and σ has zero entropy.

Theorem 8

Let X be a compact Kähler space of dimension ≤ 2. Assume that X admits a wild automorphism σ. Then X is a complex torus, and σ has zero entropy.

Proof.

Note that X is smooth and $\kappa(X) \leq 0$.
When $\dim X = 1$, X is an elliptic curve.
When $\dim X = 2$, we have

□

Theorem 8

Let X be a compact Kähler space of dimension ≤ 2. Assume that X admits a wild automorphism σ. Then X is a complex torus, and σ has zero entropy.

Proof.

Note that X is smooth and $\kappa(X) \leq 0$.
When $\dim X = 1$, X is an elliptic curve.
When $\dim X = 2$, we have

- $\kappa(X) = -\infty$: X admits a smooth fibration $f\colon X \to Y$, with fibres F smooth rational curve and Y an elliptic curve. But then F has ample $-K_F$, a contradiction.

□

Theorem 8

Let X be a compact Kähler space of dimension ≤ 2. Assume that X admits a wild automorphism σ. Then X is a complex torus, and σ has zero entropy.

Proof.

Note that X is smooth and $\kappa(X) \leq 0$.
When $\dim X = 1$, X is an elliptic curve.
When $\dim X = 2$, we have

- $\kappa(X) = -\infty$: X admits a smooth fibration $f\colon X \to Y$, with fibres F smooth rational curve and Y an elliptic curve. But then F has ample $-K_F$, a contradiction.
- $\kappa(X) = 0$: X is either a complex torus or a hyperelliptic surface. Then X is a complex torus.

□

Proposition 9

Let X be a compact complex surface which is not Kähler. Suppose that X has a wild automorphism σ. Then X is an Inoue surface of type $S_M^{(+)}$, and σ has zero entropy.

Proposition 9

Let X be a compact complex surface which is not Kähler. Suppose that X has a wild automorphism σ. Then X is an Inoue surface of type $S_M^{(+)}$, and σ has zero entropy.

Proof.

By a result of Cantat, any automorphism of a non-Kähler surface has zero entropy. The surface X has to be minimal.

class of the surface X	$\kappa(X)$	$a(X)$	$b_1(X)$	$b_2(X)$	$e(X)$
surfaces of class VII	$-\infty$	$0, 1$	1	≥ 0	≥ 0
primary Kodaira surfaces	0	1	3	4	0
secondary Kodaira surfaces	0	1	1	0	0
properly elliptic surfaces	1	1			≥ 0

Finally, we conclude that X must be an Inoue surface of type $S_M^{(+)}$. □

Inoue surfaces

An Inoue surface X is a compact complex surface obtained from $W := \mathbb{H} \times \mathbb{C}$ as a quotient by an infinite discrete group, where $\mathbb{H}$ is the upper half complex plane. Inoue surfaces are minimal surfaces in class VII, contain no curve, and have the following numerical invariants:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0.$$

There are three families of Inoue surfaces: S_M, $S_M^{(+)}$, and $S_M^{(-)}$.

Let $M = (m_{i,j}) \in \mathrm{SL}_3(\mathbb{Z})$ be a matrix with eigenvalues $\alpha, \beta, \overline{\beta}$ such that $\alpha > 1$ and $\beta \neq \overline{\beta}$. Take $(a_1, a_2, a_3)^T$ to be a real eigenvector of M corresponding to α, and $(b_1, b_2, b_3)^T$ an eigenvector corresponding to β. Let G_M be the group of automorphisms of W generated by

$$g_0(w, z) = (\alpha w, \beta z),$$
$$g_i(w, z) = (w + a_i, z + b_i), \quad i = 1, 2, 3,$$

which satisfy these conditions

$$g_0 g_i g_0^{-1} = g_1^{m_{i,1}} g_2^{m_{i,2}} g_3^{m_{i,3}},$$
$$g_i g_j = g_j g_i, \quad i, j = 1, 2, 3.$$

Note that $G_M = G_1 \rtimes G_0$ where

$$G_1 = \{g_1^{n_1} g_2^{n_2} g_3^{n_3} \mid n_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3 \quad \text{and} \quad G_0 = \langle g_0 \rangle \simeq \mathbb{Z}.$$

It can be shown that the action of G_M on W is free and properly discontinuous.

Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix with two real eigenvalues α and $1/\alpha$ with $\alpha > 1$. Let $(a_1, a_2)^T$ and $(b_1, b_2)^T$ be real eigenvectors of M corresponding to α and $1/\alpha$, respectively, and fix integers $p_1, p_2, r\,(r \neq 0)$ and a complex number τ. Define $(c_1, c_2)^T$ to be the solution of the following equation

$$(I - M)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1 a_2 - b_2 a_1}{r}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2} m_{i,1}(m_{i,1} - 1) a_1 b_1 + \frac{1}{2} m_{i,2}(m_{i,2} - 1) a_2 b_2 + m_{i,1} m_{i,2} b_1 a_2, \quad i = 1, 2.$$

Let $G_M^{(+)}$ be the group of analytic automorphisms of $W = \mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0 &\colon (w, z) \longmapsto (\alpha w, z + \tau), \\ g_i &\colon (w, z) \longmapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2, \\ g_3 &\colon (w, z) \longmapsto \left(w, z + \frac{b_1 a_2 - b_2 a_1}{r}\right). \end{aligned}$$

The action of $G_M^{(+)}$ is free and properly discontinuous.

Type $S_M^{(-)}$

Let $M \in \mathrm{GL}_2(\mathbb{Z})$ be a matrix with two real eigenvalues α and $-1/\alpha$ with $\alpha > 1$. Let $(a_1, a_2)^T$ and $(b_1, b_2)^T$ be real eigenvectors of M corresponding to α and $1/\alpha$, respectively, and fix integers $p_1, p_2, r\ (r \neq 0)$ and a complex number τ. Define $(c_1, c_2)^T$ to be the solution of the following equation

$$-(I+M)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1a_2 - b_2a_1}{r}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2}m_{i,1}(m_{i,1}-1)a_1b_1 + \frac{1}{2}m_{i,2}(m_{i,2}-1)a_2b_2 + m_{i,1}m_{i,2}b_1a_2, \quad i = 1, 2.$$

Let $G_M^{(-)}$ be the group of analytic automorphisms of $W = \mathbb{H} \times \mathbb{C}$ generated by

$$g_0 \colon (w, z) \longmapsto (\alpha w, -z),$$
$$g_i \colon (w, z) \longmapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2,$$
$$g_3 \colon (w, z) \longmapsto \left(w, z + \frac{b_1a_2 - b_2a_1}{r}\right).$$

The action of $G_M^{(-)}$ is free and properly discontinuous.

Let $M \in \mathrm{GL}_n(\mathbb{Z})$ be a diagonalisable matrix where $n = 2$ or 3. Assume that M has either

- two real eigenvalues α $(\neq \pm 1)$ and $1/\alpha$ or $-1/\alpha$, when $n = 2$; or
- three eigenvalues α $(\neq \pm 1)$, β and $\overline{\beta}$ $(\beta \neq \overline{\beta})$, when $n = 3$.

Denote

$$\Gamma := \{N \in \mathrm{GL}_n(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\}.$$

Then $\Gamma \simeq U \times \mathbb{Z}$ where U is a finite group. In particular, if we denote by $M^{\mathbb{Z}}$ the subgroup of Γ generated by M, then the quotient $\Gamma/M^{\mathbb{Z}}$ is finite.

Let $M \in \mathrm{GL}_n(\mathbb{Z})$ be a diagonalisable matrix where $n = 2$ or 3. Assume that M has either

- two real eigenvalues α $(\neq \pm 1)$ and $1/\alpha$ or $-1/\alpha$, when $n = 2$; or
- three eigenvalues α $(\neq \pm 1)$, β and $\overline{\beta}$ $(\beta \neq \overline{\beta})$, when $n = 3$.

Denote

$$\Gamma := \{N \in \mathrm{GL}_n(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\}.$$

Then $\Gamma \simeq U \times \mathbb{Z}$ where U is a finite group. In particular, if we denote by $M^{\mathbb{Z}}$ the subgroup of Γ generated by M, then the quotient $\Gamma/M^{\mathbb{Z}}$ is finite.

Theorem 10

Let X be an Inoue surface.

- If X is of type S_M or $S_M^{(-)}$, then the automorphism group $\mathrm{Aut}(X)$ is finite.

Let $M \in \mathrm{GL}_n(\mathbb{Z})$ be a diagonalisable matrix where $n = 2$ or 3. Assume that M has either

- two real eigenvalues α $(\neq \pm 1)$ and $1/\alpha$ or $-1/\alpha$, when $n = 2$; or
- three eigenvalues α $(\neq \pm 1)$, β and $\overline{\beta}$ $(\beta \neq \overline{\beta})$, when $n = 3$.

Denote

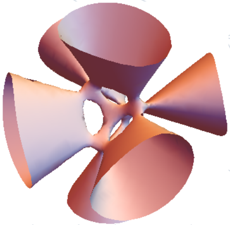
$$\Gamma := \{N \in \mathrm{GL}_n(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\}.$$

Then $\Gamma \simeq U \times \mathbb{Z}$ where U is a finite group. In particular, if we denote by $M^{\mathbb{Z}}$ the subgroup of Γ generated by M, then the quotient $\Gamma/M^{\mathbb{Z}}$ is finite.

Theorem 10

Let X be an Inoue surface.

- If X is of type S_M or $S_M^{(-)}$, then the automorphism group $\mathrm{Aut}(X)$ is finite.
- If X is of type $S_M^{(+)}$, then the neutral connected component $\mathrm{Aut}_0(X)$ of the automorphism group $\mathrm{Aut}(X)$ is isomorphic to $\mathbb{C}^*$ and the group of components $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is finite.



Dimension 3

Dimension 3

A weak Calabi-Yau manifold X is a complex projective manifold with torsion canonical divisor and finite fundamental group.
In particular, $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{Pic}^0(X)$ is trivial.

A weak Calabi-Yau manifold X is a complex projective manifold with torsion canonical divisor and finite fundamental group.
In particular, $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{Pic}^0(X)$ is trivial.

Theorem 11

Let X be a compact Kähler space of dimension three, and let σ be a wild automorphism of X. Then

A weak Calabi-Yau manifold X is a complex projective manifold with torsion canonical divisor and finite fundamental group.
In particular, $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{Pic}^0(X)$ is trivial.

Theorem 11

Let X be a compact Kähler space of dimension three, and let σ be a wild automorphism of X. Then

1. X is either a complex torus or a weak Calabi-Yau threefold;

A weak Calabi-Yau manifold X is a complex projective manifold with torsion canonical divisor and finite fundamental group.
In particular, $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{Pic}^0(X)$ is trivial.

Theorem 11

Let X be a compact Kähler space of dimension three, and let σ be a wild automorphism of X. Then

1. X is either a complex torus or a weak Calabi-Yau threefold;
2. σ has zero entropy.

Weak Calabi-Yau threefolds

Now we consider a weak Calabi-Yau threefold X.
By a result of Miyaoka(1987), we have $c_2(X) \cdot D \geq 0$ for each nef Cartier divisor D on X.
Moreover, by Kobayashi(1987), $c_2(X) \neq 0$, and thus, $c_2(X) \cdot H > 0$ for every ample Cartier divisor H.

Proposition 12

Let X be a weak Calabi-Yau threefold, and let $c_2(X)$ be the second Chern class of X.
Assume that either

Weak Calabi-Yau threefolds

Now we consider a weak Calabi-Yau threefold X.
By a result of Miyaoka(1987), we have $c_2(X) \cdot D \geq 0$ for each nef Cartier divisor D on X.
Moreover, by Kobayashi(1987), $c_2(X) \neq 0$, and thus, $c_2(X) \cdot H > 0$ for every ample Cartier divisor H.

Proposition 12

Let X be a weak Calabi-Yau threefold, and let $c_2(X)$ be the second Chern class of X. Assume that either

1. $c_2(X) \cdot D > 0$ for every non-torsion nef Cartier divisor D on X; or

Now we consider a weak Calabi-Yau threefold X.
By a result of Miyaoka(1987), we have $c_2(X) \cdot D \geq 0$ for each nef Cartier divisor D on X.
Moreover, by Kobayashi(1987), $c_2(X) \neq 0$, and thus, $c_2(X) \cdot H > 0$ for every ample Cartier divisor H.

Proposition 12

Let X be a weak Calabi-Yau threefold, and let $c_2(X)$ be the second Chern class of X. Assume that either

1. $c_2(X) \cdot D > 0$ for every non-torsion nef Cartier divisor D on X; or
2. there exists a non-torsion semi-ample Cartier divisor D on X such that $c_2(X) \cdot D = 0$.

Weak Calabi-Yau threefolds

Now we consider a weak Calabi-Yau threefold X.
By a result of Miyaoka(1987), we have $c_2(X) \cdot D \geq 0$ for each nef Cartier divisor D on X.
Moreover, by Kobayashi(1987), $c_2(X) \neq 0$, and thus, $c_2(X) \cdot H > 0$ for every ample Cartier divisor H.

Proposition 12

Let X be a weak Calabi-Yau threefold, and let $c_2(X)$ be the second Chern class of X.
Assume that either

1. $c_2(X) \cdot D > 0$ for every non-torsion nef Cartier divisor D on X; or
2. there exists a non-torsion semi-ample Cartier divisor D on X such that $c_2(X) \cdot D = 0$.

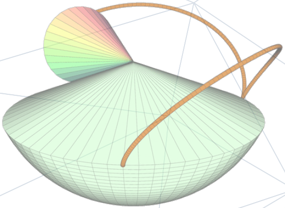
Now we consider a weak Calabi-Yau threefold X.
By a result of Miyaoka(1987), we have $c_2(X) \cdot D \geq 0$ for each nef Cartier divisor D on X.
Moreover, by Kobayashi(1987), $c_2(X) \neq 0$, and thus, $c_2(X) \cdot H > 0$ for every ample Cartier divisor H.

Proposition 12

Let X be a weak Calabi-Yau threefold, and let $c_2(X)$ be the second Chern class of X.
Assume that either

1. $c_2(X) \cdot D > 0$ for every non-torsion nef Cartier divisor D on X; or
2. there exists a non-torsion semi-ample Cartier divisor D on X such that $c_2(X) \cdot D = 0$.

Then X has no wild automorphism.



Thanks for your attention!