Potential density of projective varieties

Topology & Geometry Seminar

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January 12, 2022

The focus of this talk will be on the density of rational points on an algebraic variety, after a finite extension of the base field.

This talk is based on the following joint work.

[JSZ21]

Potential density of projective varieties having an int-amplified endomorphism,

Jia Jia, Takahiro Shibata and De-Qi Zhang,

arXiv:2108.11595

Potential density of varieties admitting int-amplified endomorphisms

K-rational points

Let K be a number field.

Let X be the subvariety of $\mathbb{A}_K^n$ defined by of polynomials with coefficients in K:

$$f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_a) = \cdots = f_m(x_1, \ldots, x_n) = 0.$$

Then a *K-rational point* on X is an n-tuple $(a_1, \ldots, a_n) \in K^n$ such that $f_1(a_1, \ldots, a_n) = \cdots = f_m(a_1, \ldots, a_n) = 0$.

Example

When $K = \mathbb{Q}$, the rational points of the unit circle of equation $x^2 + y^2 = 1$ are the pairs of rational numbers

$$\left(\pm \frac{a}{c}, \pm \frac{b}{c} \right)$$

where (a, b, c) is a Pythagorean triple.

For a scheme X over a field k, its *k-rational points* (denoted by $X(k)$) is the set of points $x \in X$ such that $k(x) := \mathcal{O}_x / \mathfrak{m}_x = k$. Equivalently, a k-rational point of X can be identified with a section of the structure morphism $X \to \operatorname{Spec} k$.

Potential density

Definition

A variety X defined over a number field K is said to satisfy *potential density* (PD) if there is a finite field extension $K \subseteq L$ such that $X_L(L)$ is Zariski dense in X_L, where $X_L := X \times_{\operatorname{Spec} K} \operatorname{Spec} L$.

Example

- Rational points of $\mathbb{P}^n$ are dense over $\mathbb{Q}$.
- Consider the curve

$$X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{Q}}.$$

Although $X(\mathbb{Q}) = \emptyset$, rational points are potentially dense. Over $\mathbb{Q}(i)$ one has

$$\begin{array}{ccccc} X & \xrightarrow{\simeq} & C = \{x^2 + y^2 = z^2\} & \xleftarrow{\simeq} & \mathbb{P}^1 \\ [x : y : z] & \longmapsto & [x : y : iz] & & \\ & & [2st : s^2 - t^2 : s^2 + t^2] & \longleftarrow\!\!\mapsfrom & [s : t] \end{array}$$

More examples

Proposition

Let $X \dashrightarrow Y$ be a dominant rational map of projective varieties over a number field. Assume that X satisfies PD, then so dose Y.
In particular, PD is a birational property.

Corollary

Unirational varieties over a number field satisfy PD.

Theorem (Chevalley-Weil)

Let $X \to Y$ be an étale morphism of proper varieties over a number field. Assume that Y satisfies PD, then so does X.

Proposition

Let A be an abelian variety over a number field K. After passing to a finite extension L/K, there is an L-rational point p on A such that $\mathbb{Z}p = \{np \mid n \in \mathbb{Z}\}$ is dense.

Corollary

Abelian varieties over a number field satisfy PD.

Non-examples

Theorem (Faltings 1983)

Let C be a curve of genus ≥ 2 over a number field K. Then $C(K)$ is finite.

Corollary

Let X be a variety with a dominant rational map $X \dashrightarrow C$ to a curve of genus ≥ 2 over a number field. Then X does not satisfy PD.

Conjecture (Lang-Bombieri)

Let X be a projective variety of general type defined over a number field. Then rational points on X are not potentially dense.

The above conjecture holds for subvarieties of abelian varieties which are of general type (Faltings).

Uniruled and rationally connected variety

Definition

A variety X is said to be *ruled* if it is birational to $U \times \mathbb{P}^1$. We say that X is *uniruled* if X is dominated by a ruled variety of the same dimension.

Definition

We say that a proper variety X over a field k is *rationally connected* if there exist a variety Y and a rational map $e: \mathbb{P}^1 \times Y \dashrightarrow X$ such that the rational map

$$\mathbb{P}^1 \times \mathbb{P}^1 \times Y \dashrightarrow X \times X, \quad (t, t', y) \longmapsto (e(t, y), e(t' y))$$

is dominant.

When k is algebraically closed of characteristic zero, if X is rationally connected, then any two closed points of X are connected by an irreducible rational curve over k.

The converse holds when k is also uncountable.

Example

- Unirational varieties, (klt) Fano varieties are rationally connected.
- Rationally connected varieties are uniruled.

Special variety and PD

Definition

An algebraic variety is *special* if it does not admit a fibration of general type in the sense of [Campana, Orbifolds, special varieties and classification theory, 2.41].

Examples

- A variety of general type is NOT special.
- A curve is special iff its genus is 0 or 1.
- A surface with no finite étale cover which dominates a positive-dimensional variety of general type, is special.
- Rationally connected varieties are special.
- Algebraic varieties with vanishing Kodaira dimension are special.

Conjcture (Campana)

Let X be a smooth projective variety defined over a number field. Then X is special iff X satisfies PD.

Int-amplified endomorphism and PD

Definition

A surjective morphism $f: X \to X$ of a projective variety is called int-amplified if there exists an ample Cartier divisor H on X such that $f^*H - H$ is ample.

Conjecture 1 (Potential density under int-amplified endomorphisms).

Let X be a projective variety defined over a number field K. Suppose that X admits an int-amplified endomorphism. Then X satisfies potential density.

Why int-amplified

- Let $f: X \to X$ be an int-amplified endomorphism of a normal projective variety X.
  either X is Q-abelian (= finite quasi-étale cover of an abelian variety $\implies \kappa = 0 \implies$ special),
  or X is uniruled:
  X is rationally connected ($\implies$ special), or
  special maximal rationally connected fibration to a lower dimension Q-abelian variety.
- We may run minimal model program equivariantly on such X (Meng and Zhang).
- Let $X = X_1 \times C$, where X_1 is any smooth projective variety and C is any smooth projective curve of genus at least 2. Such X does not satisfy PD.
  Let f be a surjective endomorphism of X. After iteration, it has the form

$$(x_1, x_2) \longmapsto (g(x_1, x_2), x_2)$$

  for some morphism $g: X_1 \times C \to X_1$ (Sano). Hence, f descends to the identity map id_C on C via the natural projection $X \to C$. Such an f is not int-amplified.
- and,

Conjecture 2 (Zariski dense orbit conjecture = ZDO).

Let X be a variety defined over an algebraically closed field k of characteristic zero and $f : X \dashrightarrow X$ a dominant rational map. Then one of the following holds:

(1) the f^*-invariant function field $k(X)^f$ is non-trivial, that is, $k(X)^f \neq k$; or

(2) there exists some $x \in X(k)$ whose f-orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is well-defined and Zariski dense.

The above conjecture holds for

- (X, f) with X being a curve (Amerik);
- (X, f) with X being a projective surface and f an endomorphism (Xie, Zhang and J.);
- (X, f) with X being an abelian variety and f an endomorphism (Ghioca and Scanlon).

Lemma 3.

Let X be a projective variety over K, $f : X \to X$ a surjective morphism, and $Z \subseteq X$ a subvariety which satisfies PD (e.g., Z is an abelian variety or unirational). If $O_f(Z)$ is Zariski dense, then X satisfies PD.

Lemma 4.

Let X be a projective variety over k and $f : X \to X$ an int-amplified endomorphism. Then $k(X)^f = k$. In particular, if ZDO holds for (X, f), then there exists some $x \in X(k)$ such that $O_f(x)$ is Zariski dense in X.

Proposition 5.

Let X be a rationally connected projective variety over K. Suppose that $\dim X \leq 3$ and X admits an int-amplified endomorphism. Then X satisfies potential density.

- either we have a Zariski dense orbit; or
- take $x \in X(\overline{K})$ with maximal $\dim(Z := \overline{O_f(x)}) = r < \dim X$.

  pick an f-fixed point $y \in X(\overline{K}) \setminus Z$, and a rational curve C connecting x and y.
  either $W := \overline{O_f(C)} = X$; or
  $\dim W = r$, then $W = Z \cup \bigcup W_i \implies y \in f^n(C) \subseteq Z$, a contradiction; or
  $r < \dim W < \dim X$, then some irreducible $W' \subseteq W$ is f-invariant and $\dim W' > r$.
  apply ZDO to $(W', f|_{W'})$ to find some w with $\dim \overline{O_f(w)} > r$.

Proposition 6.

Let X be a non-uniruled projective variety over K. Suppose that X admits an int-amplified endomorphism. Then X satisfies PD.

After lifting to normalisation, X is Q-abelian (Sheng) $\implies \exists$ dense orbit.

Theorem 7.

Let X be a normal projective variety over K with at worst $\mathbb{Q}$-factorial klt singularities. Suppose that $\dim X \leq 3$ and X admits an int-amplified endomorphism. Then X satisfies PD.

Essentially, only need to consider uniruled but not rationally connected threefold, with K_X not being pseudo-effective.

Then run EMMP which end with a Fano contraction: base and general fibre have dimension 1 or 2.

Zariski density of points with maximal arithmetic degree

Let X be a projective variety and $f: X \to X$ a surjective morphism.

Definition

The *first dynamic degree* of f is the limit

$$d_1(f) := \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n},$$

where H is an ample Cartier divisor on X.

Definition

Fix a (logarithm) height function $h_H \geq 1$ associated to an ample Cartier divisor H on X. For $x \in X(\overline{K})$, the *arithmetic degree* of f at x is the limit

$$\alpha_f(x) := \lim_{n\to\infty} h_H(f^n(x))^{1/n}.$$

Remark (Kawaguchi and Silverman; Matsuzawa)

The inequality $1 \leq \alpha_f(x) \leq d_1(f)$ holds for all $x \in X(\overline{K})$.

Conjecture (sAND; Matsuzawa, Meng, Shibata and Zhang,)

Let X be a projective variety over a number field K, $f : X \to X$ a surjective morphism, and $d > 0$ a positive integer. Then the set

$$Z_f(d) := \{x \in X(\overline{K}) \mid [K(x) : K] \leq d, \alpha_f(x) < d_1(f)\}$$

is not Zariski dense.

DR

Let L be an intermediate field: $K \subseteq L \subseteq \overline{K}$. We say that (X, f) has *densely many L-rational points with the maximal arithmetic degree (DR)$_L$* if there is a subset $S \subseteq X(L)$ satisfying the following conditions:

(1) S is Zariski dense in X_L;

(2) the equality $\alpha_f(x) = d_1(f)$ holds for all $x \in S$; and

(3) $O_f(x_1) \cap O_f(x_2) = \emptyset$ for any pair of distinct points $x_1, x_2 \in S$.

We say that (X, f) satisfies *(DR)* if there is a finite field extension $K \subseteq L$ $(\subseteq \overline{K})$ such that (X, f) satisfies *(DR)$_L$*.

Question (Sano and Shibata)

Let X be a projective variety over a number field K satisfying PD and $f : X \dashrightarrow X$ a dominant rational map over K with $d_1(f) > 1$. Does (X, f) satisfy *(DR)*?

Remark

When $d_1(f) = 1$, all points have maximal arithmetic degree. But the question is not trivial.

Let K be a number field. Let X be a projective variety over K and $f : X \to X$ a surjective morphism with $d_1(f) > 1$.

Theorem (Sano and Shibata)

- If X is unirational, then (X, f) satisfies *(DR)*$_K$.
- If X is abelian, then (X, f) satisfies *(DR)*.
- If X is a smooth projective surface satisfying PD, then (X, f) satisfies *(DR)*.

Main results

Theorem 8.

Let X be a normal projective surface over K satisfying PD, and let $f : X \to X$ be a surjective morphism with $d_1(f) > 1$. Then (X, f) satisfies (DR).

Theorem 9.

Let X be a rationally connected smooth projective threefold over K, and let $f : X \to X$ be an int-amplified endomorphism. Then (X, f) satisfies (DR).

Questions?